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The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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Gauss Contest

Grade 8

(The Grade 7 Contest is on the reverse side)

Wednesday, May 13, 2015
(in North America and South America)

Thursday, May 14, 2015
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 1 hour

Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.

Instructions

1. Do not open the contest booklet until you are told to do so.
2. You may use rulers, compasses and paper for rough work.
3. Be sure that you understand the coding system for your answer sheet. If you are not sure, ask your teacher to explain it.
4. This is a multiple-choice test. Each question is followed by five possible answers marked **A**, **B**, **C**, **D**, and **E**. Only one of these is correct. When you have made your choice, enter the appropriate letter for that question on your answer sheet.
5. Scoring: Each correct answer is worth 5 in Part A, 6 in Part B, and 8 in Part C.
   There is *no penalty* for an incorrect answer.
   Each unanswered question is worth 2, to a maximum of 10 unanswered questions.
6. Diagrams are *not* drawn to scale. They are intended as aids only.
7. When your supervisor instructs you to start, you will have *sixty* minutes of working time.

The name, school and location of some top-scoring students will be published on the Web site, cemc.uwaterloo.ca. You will also be able to find copies of past Contests and excellent resources for enrichment, problem solving and contest preparation.

Grade 8

Scoring:	There is *no penalty* for an incorrect answer. Each unanswered question is worth 2, to a maximum of 10 unanswered questions.

Part A: Each correct answer is worth 5.

1. The value of $1000 + 200 - 10 + 1$ is

   (A) 1191 **(B)** 1190 **(C)** 1189 **(D)** 1209 **(E)** 1211

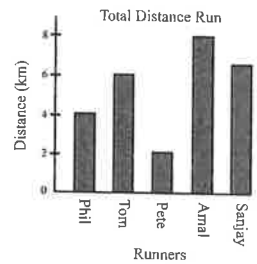
2. What time is it 45 minutes after 10:20?

   (A) 11:00 **(B)** 9:35 **(C)** 11:15 **(D)** 10:55 **(E)** 11:05

3. Which of the following is closest to 5 cm?

   (A) The length of a full size school bus
   (B) The height of a picnic table
   (C) The height of an elephant
   (D) The length of your foot
   (E) The length of your thumb

4. The graph shows the total distance that each of five runners ran during a one-hour training session. Which runner ran the median distance?

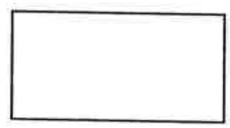


   (A) Phil **(B)** Tom **(C)** Pete
   (D) Amal **(E)** Sanjay

5. If $x + 3 = 10$, what is the value of $5x + 15$?

   (A) 110 **(B)** 35 **(C)** 80 **(D)** 27 **(E)** 50

6. A rectangle has a perimeter of 42 and a width of 4. What is its length?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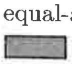
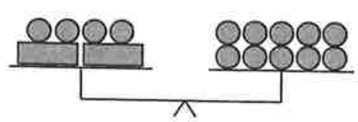
   (A) 19 **(B)** 17 **(C)** 34
   (D) 21 **(E)** 38

7. The equal-arm scale shown is balanced.
   One  has the same mass as

   (A) 
   (B) 
   (C) 
   (D) 
   (E) 



8. At the beginning of the summer, Aidan was 160 cm tall. At the end of the summer, he measured his height again and discovered that it had increased by 5%. Measured in cm, what was his height at the end of summer?

   (A) 168 **(B)** 165 **(C)** 160.8 **(D)** 172 **(E)** 170

9. If $x = 4$ and $y = 2$, which of the following expressions gives the smallest value?

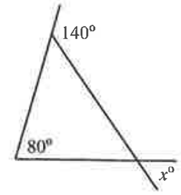
   (A) $x + y$ **(B)** xy **(C)** $x - y$ **(D)** $x \div y$ **(E)** $y \div x$

10. The number represented by □ so that $\frac{1}{2} + \frac{1}{4} = \frac{\square}{12}$ is

(A) 3 (B) 12 (C) 9 (D) 6 (E) 15

Part B: Each correct answer is worth 6.

11. In the diagram, the value of x is

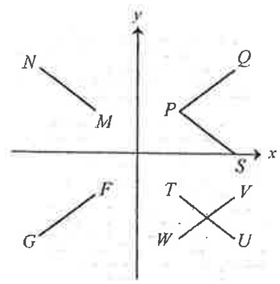


(A) 40 (B) 50 (C) 60
(D) 70 (E) 80

12. Zara's bicycle tire has a circumference of 1.5 m. If Zara travels 900 m on her bike, how many full rotations will her tire make?

(A) 900 (B) 1350 (C) 600 (D) 450 (E) 1200

13. In the graph shown, which of the following represents the image of the line segment PQ after a reflection across the x-axis?



(A) PS (B) TU (C) MN
(D) WV (E) FG

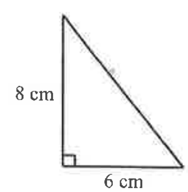
14. Carolyn has a \$5 bill, a \$10 bill, a \$20 bill, and a \$50 bill in her wallet. She closes her eyes and removes one of the four bills from her wallet. What is the probability that the total value of the three bills left in her wallet is greater than \$70?

(A) 0.5 (B) 0.25 (C) 0.75 (D) 1 (E) 0

15. Two puppies, Walter and Stanley, are growing at different but constant rates. Walter's mass is 12 kg and he is growing at a rate of 2 kg/month. Stanley's mass is 6 kg and he is growing at a rate of 2.5 kg/month. What will Stanley's mass be when it is equal to Walter's?

(A) 24 kg (B) 28 kg (C) 32 kg (D) 36 kg (E) 42 kg

16. There is a square whose perimeter is the same as the perimeter of the triangle shown. The area of that square is



(A) $12.25\,\text{cm}^2$ (B) $196\,\text{cm}^2$ (C) $49\,\text{cm}^2$
(D) $36\,\text{cm}^2$ (E) $144\,\text{cm}^2$

17. When expressed as a repeating decimal, the fraction $\frac{1}{7}$ is written as 0.142857142857... (The 6 digits 142857 continue to repeat.) The digit in the third position to the right of the decimal point is 2. In which one of the following positions to the right of the decimal point will there also be a 2?

(A) 119^{th} (B) 121^{st} (C) 123^{rd} (D) 125^{th} (E) 126^{th}

18. The operation Δ is defined so that $a\Delta b = a \times b + a + b$. For example, $2\Delta 5 = 2 \times 5 + 2 + 5 = 17$. If $p\Delta 3 = 39$, the value of p is

(A) 13 **(B)** 12 **(C)** 9 **(D)** 10.5 **(E)** 18

19. There are 3 times as many boys as girls in a room. If 4 boys and 4 girls leave the room, then there will be 5 times as many boys as girls in the room. In total, how many boys and girls were in the room originally?

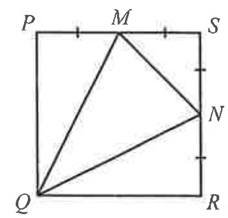
(A) 15 **(B)** 20 **(C)** 24 **(D)** 32 **(E)** 40

20. A rectangle has side lengths 3 and 4. One of its vertices is at the point $(1,2)$. Which of the following *could not* be the coordinates of one of its other vertices?

(A) $(-3,-1)$ **(B)** $(1,-5)$ **(C)** $(5,-1)$ **(D)** $(-2,6)$ **(E)** $(1,-1)$

Part C: Each correct answer is worth 8.

21. In square $PQRS$, M is the midpoint of PS and N is the midpoint of SR. If the area of $\triangle SMN$ is 18, then the area of $\triangle QMN$ is



(A) 36 **(B)** 72 **(C)** 90

(D) 48 **(E)** 54

22. Exactly 120 tickets were sold for a concert. The tickets cost \$12 each for adults, \$10 each for seniors, and \$6 each for children. The number of adult tickets sold was equal to the number of child tickets sold. Given that the total revenue from the ticket sales was \$1100, the number of senior tickets sold was

(A) 110 **(B)** 20 **(C)** 40 **(D)** 2 **(E)** 18

23. The list of integers $4, 4, x, y, 13$ has been arranged from least to greatest. How many different possible ordered pairs (x, y) are there so that the average (mean) of these 5 integers is itself an integer?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

24. Two joggers each run at their own constant speed and in opposite directions from one another around an oval track. They meet every 36 seconds. The first jogger completes one lap of the track in a time that, when measured in seconds, is a number (not necessarily an integer) between 80 and 100. The second jogger completes one lap of the track in a time, t seconds, where t is a positive integer. The product of the smallest and largest possible integer values of t is

(A) 3705 **(B)** 3762 **(C)** 2816 **(D)** 3640 **(E)** 3696

25. The *alternating sum* of the digits of 63 195 is $6 - 3 + 1 - 9 + 5 = 0$. In general, the alternating sum of the digits of a positive integer is found by taking its leftmost digit, subtracting the next digit to the right, adding the next digit to the right, then subtracting, and so on. A positive integer is divisible by 11 exactly when the alternating sum of its digits is divisible by 11. For example, 63 195 is divisible by 11 since the alternating sum of its digits is equal to 0, and 0 is divisible by 11. Similarly, 92 807 is divisible by 11 since the alternating sum of its digits is 22, but 60 432 is not divisible by 11 since the alternating sum of its digits is 9.

Lynne forms a 7-digit integer by arranging the digits $1, 2, 3, 4, 5, 6, 7$ in random order. What is the probability that the integer is divisible by 11?

(A) $\frac{1}{35}$ **(B)** $\frac{5}{42}$ **(C)** $\frac{3}{35}$ **(D)** $\frac{1}{42}$ **(E)** $\frac{4}{35}$

Grade 8

1. Evaluating, $1000 + 200 - 10 + 1 = 1200 - 10 + 1 = 1190 + 1 = 1191$.

   ANSWER: (A)

2. Since there are 60 minutes in an hour, then 40 minutes after 10:20 it is 11:00.
   Therefore, 45 minutes after 10:20 it is 11:05.

   ANSWER: (E)

3. Of the possible answers, the length of your thumb is closest to 5 cm.

   ANSWER: (E)

4. Reading from the graph, Phil ran 4 km, Tom ran 6 km, Pete ran 2 km, Amal ran 8 km, and Sanjay ran 7 km.
   Ordering these distances from least to greatest, we get Pete ran 2 km, Phil ran 4 km, Tom ran 6 km, Sanjay ran 7 km, and Amal ran 8 km.
   In this ordered list of 5 distances, the median distance is in the middle, the third greatest.
   Therefore, Tom ran the median distance.

   ANSWER: (B)

5. *Solution 1*
   Since $x + 3 = 10$, then $x = 10 - 3 = 7$.
   When $x = 7$, the value of $5x + 15$ is $5(7) + 15 = 35 + 15 = 50$.

   Solution 2
   When multiplying $x + 3$ by 5, we get $5 \times (x + 3) = 5 \times x + 5 \times 3 = 5x + 15$.
   Since $x + 3 = 10$, then $5 \times (x + 3) = 5 \times 10 = 50$.
   Therefore, $5 \times (x + 3) = 5x + 15 = 50$.

   ANSWER: (E)

6. The two equal widths, each of length 4, contribute $2 \times 4 = 8$ to the perimeter of the rectangle.
   The two lengths contribute the remaining $42 - 8 = 34$ to the perimeter.
   Since the two lengths are equal, they each contribute $34 \div 2 = 17$ to the perimeter.
   Therefore, the length of the rectangle is 17.

   ANSWER: (B)

7. To begin, there are 4 circles and 2 rectangles on the left arm, balanced by 10 circles on the right arm.
   If we remove 4 circles from each side of the equal-arm scale, the scale will remain balanced (since we are removing the same mass from each side).
   That is, the 2 rectangles that will remain on the left arm are equal in mass to the 6 circles that will remain on the right arm.
   Since 2 rectangles are equal in mass to 6 circles, then 1 rectangle has the same mass as 3 circles.

   ANSWER: (B)

8. *Solution 1*
   A 5% increase in 160 is equal to $\frac{5}{100} \times 160$ or $0.05 \times 160 = 8$.
   Therefore, Aidan's height increased by 8 cm over the summer.
   His height at the end of the summer was $160 + 8 = 168$ cm.

   Solution 2
   Since Aidan's 160 cm height increased by 5%, then his height at the end of the summer was $(1 + \frac{5}{100}) \times 160 = (1 + 0.05) \times 160 = 1.05 \times 160 = 168$ cm.

   ANSWER: (A)

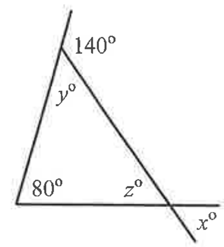
9. When $x = 4$ and $y = 2$, $x + y = 4 + 2 = 6$, $xy = 4 \times 2 = 8$, $x - y = 4 - 2 = 2$, $x \div y = 4 \div 2 = 2$, and $y \div x = 2 \div 4 = \frac{1}{2}$.
Therefore, the expression which gives the smallest value when $x = 4$ and $y = 2$ is $y \div x$.
ANSWER: (E)

10. *Solution 1*
Evaluating using a denominator of 12, $\frac{1}{2} + \frac{1}{4} = \frac{6}{12} + \frac{3}{12} = \frac{9}{12}$ and so the number represented by $\square$ is 9.

Solution 2
Since $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$ and $\frac{3}{4} = \frac{9}{12}$, then $\frac{1}{2} + \frac{2}{4} = \frac{9}{12}$.
The number represented by $\square$ is 9.

ANSWER: (C)

11. Straight angles measure 180°.
Therefore, $y° + 140° = 180°$, and so $y = 180 - 140 = 40$.
The three interior angles of any triangle add to 180°.
Thus, $40° + 80° + z° = 180°$, and so $z = 180 - 40 - 80 = 60$.
Opposite angles have equal measures.
Since the angle measuring $z°$ is opposite the angle measuring $x°$, then $x = z = 60$.



ANSWER: (C)

12. Since Zara's bicycle tire has a circumference of 1.5 m, then each full rotation of the tire moves the bike 1.5 m forward.
If Zara travels 900 m on her bike, then her tire will make $900 \div 1.5 = 600$ full rotations.
ANSWER: (C)

13. To find the image of PQ, we reflect points P and Q across the x-axis, then join them.
Since P is 3 units above the x-axis, then the reflection of P across the x-axis is 3 units below the x-axis at the same x-coordinate.
That is, point T is the image of P after it is reflected across the x-axis.
Similarly, after a reflection across the x-axis, the image of point Q will be 6 units below the x-axis but have the same x-coordinate as Q.
That is, point U is the image of Q after it is reflected across the x-axis.
Therefore, the line segment TU is the image of PQ after it is reflected across the x-axis.
ANSWER: (B)

14. In the table below, we determine the total value of the three bills that remain in Carolyn's wallet when each of the four bills is removed.

Bill Removed	Sum of the Bills Remaining
\$5	\$10+\$20+\$50=\$80
\$10	\$5+\$20+\$50=\$75
\$20	\$5+\$10+\$50=\$65
\$50	\$5+\$10+\$20=\$35

It is equally likely that any one of the four bills is removed from the wallet and therefore any of the four sums of the bills remaining in the wallet is equally likely.
Of the four possible sums, \$80, \$75, \$65, and \$35, two are greater than \$70.
Therefore, the probability that the total value of the three bills left in Carolyn's wallet is greater than \$70, is $\frac{2}{4}$ or 0.5.

ANSWER: (A)

15. In the table below, we list the mass of each dog at the end of each month.

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Walter's mass (in kg)	12	14	16	18	20	22	24	26	28	30	32	34	36
Stanley's mass (in kg)	6	8.5	11	13.5	16	18.5	21	23.5	26	28.5	31	33.5	36

After 12 months have passed, Stanley's mass is 36 kg and is equal to Walter's mass.
(Note that since Stanley's mass is increasing at a greater rate than Walter's each month, this is the only time that the two dogs will have the same mass.)

ANSWER: (D)

16. First, we must determine the perimeter of the given triangle.
Let the unknown side length measure x cm.
Since the triangle is a right-angled triangle, then by the Pythagorean Theorem we get $x^2 = 8^2 + 6^2$ or $x^2 = 64 + 36 = 100$ and so $x = \sqrt{100} = 10$ (since $x > 0$).
Therefore the perimeter of the triangle is $10+8+6 = 24$ cm and so the perimeter of the square is also 24 cm.
Since the 4 sides of the square are equal in length, then each measures $\frac{24}{4} = 6$ cm.
Thus, the area of the square is $6 \times 6 = 36$ cm^2.

ANSWER: (D)

17. Since the number of digits that repeat is 6, then the digits 142857 begin to repeat again after 120 digits (since $120 = 6 \times 20$).
That is, the 121^{st} digit is a 1, the 122^{nd} digit is a 4, and the 123^{rd} digit is a 2.

ANSWER: (C)

18. Using the definition of Δ, we see that $p\Delta 3 = p \times 3 + p + 3 = 3p + p + 3 = 4p + 3$.
Since $p\Delta 3 = 39$, then $4p + 3 = 39$ or $4p = 39 - 3 = 36$ and so $p = \frac{36}{4} = 9$.

ANSWER: (C)

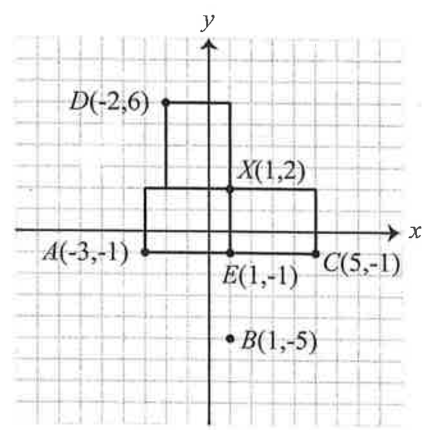
19. *Solution 1*
Originally there are 3 times as many boys as girls, so then for every 3 boys there is 1 girl and $3 + 1 = 4$ children in the room.
That is, the number of boys in the room is $\frac{3}{4}$ of the number of children in the room.
Next we consider each of the 5 possible answers, in turn, to determine which represents the total number of children in the room originally.
If the original number of children in the room is 15 (as in answer (A)), the number of boys is $\frac{3}{4} \times 15 = \frac{45}{4} = 11.25$.
Since it is not possible to have 11.25 boys in the room, then we know that 15 is not the correct answer.
If the original number of children in the room is 20 (as in answer (B)), the number of boys is $\frac{3}{4} \times 20 = \frac{60}{4} = 15$.
If the number of boys in the room was originally 15, then the number of girls was $20 - 15 = 5$.
Next we must check if there will be 5 times as many boys as girls in the room once 4 boys and 4 girls leave the room.
If 4 boys leave the room, there are 11 boys remaining. If 4 girls leave the room, there is 1 girl remaining and since there are not 5 times as many boys as girls, then 20 is not the correct answer.
If the original number of children in the room is 24 (as in answer (C)), the number of boys is $\frac{3}{4} \times 24 = \frac{72}{4} = 18$.
If the number of boys in the room was originally 18, then the number of girls was $24 - 18 = 6$.

If 4 boys leave the room, there are 14 boys left and if 4 girls leave the room, then there are 2 girls left.
Since there are not 5 times as many boys as girls, then 24 is not the correct answer.
If the original number of children in the room is 32 (as in answer (D)), the number of boys is $\frac{3}{4} \times 32 = \frac{96}{4} = 24$.
If the number of boys in the room was originally 24, then the number of girls was $32 - 24 = 8$.
If 4 boys leave the room, there are 20 left and if 4 girls leave the room, then there are 4 left.
Since there are 5 times as many boys as girls, then we know that the original number of children is 32.
(Note: We may check that the final answer, 40, gives 30 boys and 10 girls originally and when 4 boys and 4 girls leave the room there are 26 boys and 6 girls which again does not represent 5 times as many boys as girls.)

Solution 2
Originally there are 3 times as many boys as girls, so if there are x girls in the room, then there are $3x$ boys.
If 4 boys leave the room, there are $3x - 4$ boys remaining.
If 4 girls leave the room, there are $x - 4$ girls remaining.
At this point, there are 5 times as many boys as girls in the room.
That is, $5 \times (x - 4) = 3x - 4$ or $5x - 20 = 3x - 4$ and so $5x - 3x = 20 - 4$ or $2x = 16$ and so $x = 8$.
Therefore, the original number of girls in the room is 8 and the original number of boys is $3 \times 8 = 24$.
The original number of students in the room is $24 + 8 = 32$.

ANSWER: (D)

20. *Solution 1*
Call the given vertex of the rectangle $(1, 2)$ point X and name each of the 5 answers to match the letters A through E, as shown.



Point $E(1, -1)$ is 3 units below point X (since their x-coordinates are equal and their y-coordinates differ by 3). Thus, $E(1, -1)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and E would be adjacent vertices of the rectangle).

Point $C(5, -1)$ is 3 units below and 4 units right of point X (since their y-coordinates differ by 3 and their x-coordinates differ by 4). Thus, $C(5, -1)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and C would be opposite vertices of the rectangle).

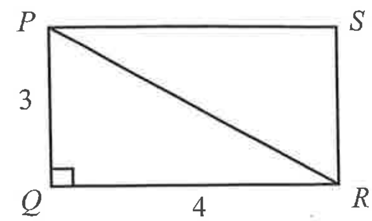
Point $A(-3, -1)$ is 3 units below and 4 units left of point X (since their y-coordinates differ by 3 and their x-coordinates differ by 4). Thus, $A(-3, -1)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and A would be opposite vertices of the rectangle).

Point $D(-2, 6)$ is 4 units above and 3 units left of point X (since their y-coordinates differ by 4 and their x-coordinates differ by 3). Thus, $D(-2, 6)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and D would be opposite vertices of the rectangle).

The only point remaining is $B(1, -5)$ and since it is possible for each of the other 4 answers to

be one of the other vertices of the rectangle, then it must be $(1, -5)$ that can not be.
Point $B(1, -5)$ is 7 units below point X (since their y-coordinates differ by 7).
How might we show that no two vertices of a 3 by 4 rectangle are 7 units apart?
(See Solution 2).

Solution 2
The distance between any two adjacent vertices of a 3 by 4 rectangle $PQRS$ is either 3 or 4 units (such as P and Q or Q and R, as shown).



The distance between any two opposite vertices of a rectangle (such as P and R) can be found using the Pythagorean Theorem.
In the right-angled triangle PQR, we get $PR^2 = 3^2 + 4^4$ or $PR^2 = 9 + 16 = 25$ and so $PR = \sqrt{25} = 5$ (since $PR > 0$).
That is, the greatest distance between any two vertices of a 3 by 4 rectangle is 5 units.
As shown and explained in Solution 1, the distance between $X(1, 2)$ and $B(1, -5)$ is 7 units.
Therefore $(1, -5)$ could not be the coordinates of one of the other vertices of a 3 by 4 rectangle having vertex $X(1, 2)$.

ANSWER: (B)

21. In square $PQRS$, $PS = SR$ and since M and N are midpoints of these sides having equal length, then $MS = SN$.
The area of $\triangle SMN$ is $\frac{1}{2} \times MS \times SN$.
Since this area equals 18, then $\frac{1}{2} \times MS \times SN = 18$ or $MS \times SN = 36$ and so $MS = SN = 6$ (since they are equal in length).
The side of the square, PS, is equal in length to $PM + MS = 6 + 6 = 12$ (since M is the midpoint of PS) and so $PS = SR = RQ = QP = 12$.
The area of $\triangle QMN$ is equal to the area of square $PQRS$ minus the combined areas of the three right-angled triangles, $\triangle SMN$, $\triangle NRQ$ and $\triangle QPM$.
Square $PQRS$ has area $PS \times SR = 12 \times 12 = 144$.
$\triangle SMN$ has area 18, as was given in the question.
$\triangle NRQ$ has area $\frac{1}{2} \times QR \times RN = \frac{1}{2} \times 12 \times 6 = 36$ (since $SN = RN = 6$).
$\triangle QPM$ has area $\frac{1}{2} \times QP \times PM = \frac{1}{2} \times 12 \times 6 = 36$.
Thus the area of $\triangle QMN$ is $144 - 18 - 36 - 36 = 54$.

ANSWER: (E)

22. Let the number of adult tickets sold be a.
Since the price for each adult ticket is \$12, then the revenue from all adult tickets sold (in dollars) is $12 \times a$ or $12a$.
Since the number of child tickets sold is equal to the number of adult tickets sold, we can let the number of child tickets sold be a, and the total revenue from all \$6 child tickets be $6a$ (in dollars).
In dollars, the combined revenue of all adult tickets and all child tickets is $12a + 6a = 18a$.

Since the total number of tickets sold is 120, and a adult tickets were sold and a child tickets were sold, then the remaining $120 - 2a$ tickets were sold to seniors.
Since the price for each senior ticket is \$10, then the revenue from all senior tickets sold (in dollars) is $10 \times (120 - 2a)$.
Thus the combined revenue from all ticket sales is $10 \times (120 - 2a) + 18a$, dollars.

The total revenue from the ticket sales was \$1100 and so $10 \times (120 - 2a) + 18a = 1100$.
Solving this equation, we get $10 \times 120 - 10 \times 2a + 18a = 1100$ or $1200 - 20a + 18a = 1100$ or $1200 - 2a = 1100$ and so $2a = 100$ or $a = 50$.
Therefore, the number of senior tickets sold for the concert was
$120 - 2a = 120 - 2(50) = 120 - 100 = 20$.

We may check that the number of tickets sold to each of the three groups gives the correct total revenue.
Since the number of adult tickets sold was equal to the number of child tickets sold which was equal to a, then 50 of each were sold.
The revenue from 50 adult tickets is $50 \times \$12 = \$600$.
The revenue from 50 child tickets is $50 \times \$6 = \$300$.
The revenue from 20 senior tickets is $20 \times \$10 = \$200$.
The total revenue from all tickets sold was $\$600 + \$300 + \$200 = \$1100$, as required.

Answer: (B)

23. The list of integers $4, 4, x, y, 13$ has been arranged from least to greatest, and so $4 \leq x$ and $x \leq y$ and $y \leq 13$.
The sum of the 5 integers is $4 + 4 + x + y + 13 = 21 + x + y$ and so the average is $\frac{21 + x + y}{5}$.
Since this average is a whole number, then $21 + x + y$ must be divisible by 5 (that is, $21 + x + y$ is a multiple of 5).
How small and how large can the sum $21 + x + y$ be?
We know that $4 \leq x$ and $x \leq y$, so the smallest that $x + y$ can be is $4 + 4 = 8$.
Since $x + y$ is at least 8, then $21 + x + y$ is at least $21 + 8 = 29$.
Using the fact that $x \leq y$ and $y \leq 13$, the largest that $x + y$ can be is $13 + 13 = 26$.
Since $x + y$ is at most 26, then $21 + x + y$ is at most $21 + 26 = 47$.
The multiples of 5 between 29 and 47 are $30, 35, 40$, and 45.
When $21 + x + y = 30$, we get $x + y = 30 - 21 = 9$.
The only ordered pair (x, y) such that $4 \leq x$ and $x \leq y$ and $y \leq 13$, and $x + y = 9$ is $(x, y) = (4, 5)$.
Continuing in this way, we determine all possible values of x and y that satisfy the given conditions in the table below.

Value of $21 + x + y$	Value of $x + y$	Ordered Pairs (x, y) with $4 \leq x$ and $x \leq y$ and $y \leq 13$
30	$30 - 21 = 9$	$(4, 5)$
35	$35 - 21 = 14$	$(4, 10), (5, 9), (6, 8), (7, 7)$
40	$40 - 21 = 19$	$(6, 13), (7, 12), (8, 11), (9, 10)$
45	$45 - 21 = 24$	$(11, 13), (12, 12)$

The number of ordered pairs (x, y) such that the average of the 5 integers $4, 4, x, y, 13$ is itself an integer is 11.

Answer: (E)

24. The two joggers meet every 36 seconds.
Therefore, the combined distance that the two joggers run every 36 seconds is equal to the total distance around one lap of the oval track, which is constant.
Thus the greater the first jogger's constant speed, the greater the distance that they run every 36 seconds, meaning the second jogger runs less distance in the same time (their combined

distance is constant) and hence the smaller the second jogger's constant speed.
Conversely, the slower the first jogger's constant speed, the less distance that they run every 36 seconds, meaning the second jogger must run a greater distance in this same time and hence the greater the second jogger's constant speed.

This tells us that if the first jogger completes one lap of the track as fast as possible, which is in 80 seconds, then the second jogger's time to complete one lap of the track is as slow as possible.
We will call this time t_{max}, the maximum possible time that it takes the second jogger to complete one lap of the track.
Similarly, if the first jogger completes one lap of the track as slowly as possible, which is in 100 seconds, then the second jogger's time to complete one lap of the track is as fast as possible.
We will call this time t_{min}, the minimum possible time that it takes the second jogger to complete one lap of the track.

Finding the value of t_{max}

Recall that t_{max} is the time it takes the second jogger to complete one lap when the first jogger completes one lap of the track in 80 seconds.
If the first jogger can complete one lap of the track in 80 seconds, then in 36 seconds of running, the first jogger will complete $\frac{36}{80} = \frac{9}{20}$ of a complete lap of the track.
In this same 36 seconds, the two joggers combined distance running is 1 lap, and so the second jogger runs $1 - \frac{9}{20} = \frac{11}{20}$ of a complete lap.
If the second jogger runs $\frac{11}{20}$ of a complete lap in 36 seconds, then the second jogger runs $\frac{20}{11} \times \frac{11}{20} = 1$ complete lap in $\frac{20}{11} \times 36 = \frac{720}{11}$ seconds. Thus, $t_{max} = \frac{720}{11} = 65.\overline{45}$ seconds.

Finding the value of t_{min}

Recall that t_{min} is the time it takes the second jogger to complete one lap when the first jogger completes one lap of the track in 100 seconds.
If the first jogger can complete one lap of the track in 100 seconds, then in 36 seconds of running, the first jogger will complete $\frac{36}{100} = \frac{9}{25}$ of a complete lap of the track.
In this same 36 seconds, the two joggers combined distance running is 1 lap, and so the second jogger runs $1 - \frac{9}{25} = \frac{16}{25}$ of a complete lap.
If the second jogger runs $\frac{16}{25}$ of a complete lap in 36 seconds, then the second jogger runs $\frac{25}{16} \times \frac{16}{25} = 1$ complete lap in $\frac{25}{16} \times 36 = \frac{900}{16}$ seconds. Thus, $t_{min} = \frac{900}{16} = 56.25$ seconds.

Determining the product of the smallest and largest integer values of t

Since the second jogger completes 1 lap of the track in at most $65.\overline{45}$ seconds, then the largest possible integer value of t is 65 seconds.
The second jogger completes 1 lap of the track in at least 56.25 seconds, so then the smallest possible integer value of t is 57 seconds.
Finally, the product of the smallest and largest integer values of t is $57 \times 65 = 3705$.

ANSWER: (A)

25. Let the alternating sum of the digits be S.
If the 7-digit integer is $abcdefg$, then $S = a - b + c - d + e - f + g$.
This sum can be grouped into the digits which contribute positively to the sum, and those which contribute negatively to the sum.
Rewriting the sum in this way, we get $S = (a + c + e + g) - (b + d + f)$.
Taking the 4 digits which contribute positively to S (there are always 4), we let $P = a+c+e+g$.
Similarly, taking the 3 digits which contribute negatively to S (there are always 3), we let $N = b + d + f$.
Thus, it follows that $S = (a + c + e + g) - (b + d + f) = P - N$.

We determine the largest possible value of S by choosing the 4 largest integers, $4, 5, 6, 7$ (in any order), to make up P, and choosing the 3 smallest integers, $1, 2, 3$ (in any order), to make up N.
That is, the largest possible alternating sum is $S = (4 + 5 + 6 + 7) - (1 + 2 + 3) = 16$.
We determine the smallest possible value of S by choosing the 4 smallest integers, $1, 2, 3, 4$ (in any order), to make up P, and choosing the 3 largest integers, $5, 6, 7$ (in any order), to make up N.
That is, the smallest possible alternating sum is $S = (1 + 2 + 3 + 4) - (5 + 6 + 7) = -8$.
Since S must be divisible by 11 (with $S \geq -8$ and $S \leq 16$), then either $S = 11$ or $S = 0$.
The sum of the first 7 positive integers is $1+2+3+4+5+6+7 = 28$, and since each of these 7 integers must contribute to either P or to N, then $P + N = 28$.

Case 1: The alternating sum of the digits is 11, or $S = 11$

If $S = 11$, then $S = P - N = 11$. If $P - N$ is 11 (an odd number), then either P is an even number and N is odd, or the opposite is true (they can't both be odd and they can't both be even).
That is, the difference between two integers is odd only if one of the integers is even and the other is odd (we say that P and N have *different parity*).
However, if one of P or N is even and the other is odd, then their sum $P + N$ is also odd.
But we know that $P + N = 28$, an even number.
Therefore, it is not possible that $S = 11$.
There are no 7-digit integers formed from the integers 1 through 7 that have an alternating digit sum of 11 and are divisible by 11.

Case 2: The alternating sum of the digits is 0, or $S = 0$

If $S = 0$, then $S = P - N = 0$ and so $P = N$.
Since $P + N = 28$, then $P = N = 14$.
We find all groups of 3 digits, chosen from the digits 1 to 7, such that their sum $N = 14$.
There are exactly 4 possibilities: $(7, 6, 1)$, $(7, 5, 2)$, $(7, 4, 3)$, and $(6, 5, 3)$.
In each of these 4 cases, the digits from 1 to 7 that were not chosen, $(2, 3, 4, 5)$, $(1, 3, 4, 6)$, $(1, 2, 5, 6)$, and $(1, 2, 4, 7)$, respectively, represent the 4 digits whose sum is $P = 14$.
We summarize this in the table below.

4 digits whose sum is $P = 14$	3 digits whose sum is $N = 14$	2 examples of 7-digit integers created from these
$2, 3, 4, 5$	$7, 6, 1$	$2736415, 3126475$
$1, 3, 4, 6$	$7, 5, 2$	$1735426, 6745321$
$1, 2, 5, 6$	$7, 4, 3$	$2714536, 5763241$
$1, 2, 4, 7$	$6, 5, 3$	$4615237, 7645231$

Consider the first row of numbers in this table above.
Each arrangement of the 4 digits $2, 3, 4, 5$ combined with each arrangement of the 3 digits $7, 6, 1$ (in the required way) gives a new 7-digit integer whose alternating digit sum is 0.
Two such arrangements are shown (you may check that $S = 0$ for each).
Since there are $4 \times 3 \times 2 \times 1 = 24$ ways to arrange the 4 digits (4 choices for the first digit, 3 choices for the second, 2 choices for the third and 1 choice for the last digit), and $3 \times 2 \times 1 = 6$ ways to arrange the 3 digits, then there are $24 \times 6 = 144$ ways to arrange the 4 digits and the 3 digits.
Each of these 144 arrangements is different from the others, and since $P = N = 14$ for each, then $S = P - N = 0$ and so each of the 144 7-digit numbers is divisible by 11.
Similarly, there are also 144 arrangements that can be formed with each of the other 3 groups of integers that are shown in the final 3 rows of the table.
That is, there are a total of $144 \times 4 = 576$ 7-digit integers (formed from the integers 1 through 7) which are divisible by 11.

The total number of 7-digit integers that can be formed from the integers 1 through 7 is equal to the total number of arrangements of the integers 1 through 7, or $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.
Therefore, when the digits 1 through 7 are each used to form a random 7-digit integer, the probability that the number formed is divisible by 11 is $\dfrac{576}{5040} = \dfrac{4}{35}$.

ANSWER: (E)